A New Horizontal Diffusion Operator

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Basic Dynamic Equations

Source: A Description of the Nonhydrostatic Regional COSMO-Model Part I: Dynamics and Numerics

The equation for mass of the water constituents:

$$\rho \frac{\mathrm{d}q^x}{\mathrm{d}t} = -\nabla \cdot J^x + I^x, \tag{1}$$

where $x \in \{dry\ air, water\ vapour, liquid\ water, ice\}$, I^x is the sources/sinks of constituent x and J^x is the diffusive flux of constituent x.

Applying the Reynolds-averaging operator to (1) yields prognostic equations for the corresponding mean values:

$$\bar{\rho} \frac{\hat{\mathrm{d}}\hat{q}^x}{\mathrm{d}t} = -\nabla \left(\bar{J}^x + F^x \right) + \bar{I}^x, \tag{2}$$

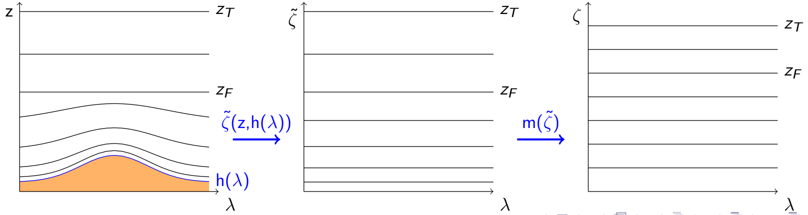
where $F^x := \overline{\rho v'' q^x}$ is the turbulent flux of constituent x.

Terrain-Following grid

Different coordinate transforms can be used to switch from the (λ, ϕ, z)-system to the $(\lambda, \phi, \tilde{\zeta})$-system. The Gal-Chen coordinate transform is one of them:

$$\tilde{\zeta}(z, h(\lambda, \phi)) = \begin{cases} \dfrac{z - h(\lambda, \phi)}{1 - \frac{h(\lambda,\phi)}{z_F}} & \text{if } z < z_F, \\ z & \text{if } z_F \leq z \leq z_T. \end{cases}$$

A monotonic function is then used to go to the (λ, ϕ, ζ)-system.



Current Situation: Subgrid Scale Turbulence Closure

Source: A Description of the Nonhydrostatic Regional COSMO Model Part II: Physical Parameterization

We express the turbulent fluxes of the moisture variables with respect to the unit base vectors of the orthogonal (λ, ϕ, z)-system :

$$F_x^1 = -\rho K \frac{1}{a\cos\phi}\left(\frac{\partial q^x}{\partial\lambda} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial q^x}{\partial\zeta}\right)$$

$$F_x^2 = -\rho K \frac{1}{a}\left(\frac{\partial q^x}{\partial\phi} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial q^x}{\partial\zeta}\right)$$

Calculated implicitly $\longrightarrow$ $F_x^3 = +\rho K \frac{1}{\sqrt{G}}\frac{\partial q^x}{\partial\zeta}$

We are interested in the horizontal components of the subrid scale flux divergence of equation (2) i.e.

$$\bar{\rho}\frac{\hat{\partial}\hat{q}^x}{\mathrm{d}t} = -\nabla F_h = \frac{1}{a\cos\phi}\left(\partial_\lambda F_x^1 + \partial_\phi(F_x^2\ \cos\phi)\right) \qquad (3)$$

Current Situation: Where is this done?

This calculation is implemented in a subroutine in src_slow_tendencies_rk.f90. The horizontal diffusion of subgrid scale turbulence is being calculated without taking into account the sloping coordinate surfaces (l3turb_metr=.FALSE.) due to the limited stability of the implemented method.

lmorg.f90 → organize_dynamics.f90 → src_runge_kutta.f90 → src_slow_tendencies_rk.f90
SUBROUTINE explicit_horizontal_diffusion

Current Situation: Stability Analysis

Let us consider the scheme which approximates the horizontal diffusion $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right)$ using finite differences and the relation: $\left.\frac{\partial u}{\partial x}\right|_z = \left.\frac{\partial u}{\partial x}\right|_\zeta - \zeta_x \left.\frac{\partial u}{\partial \zeta}\right|_x$ (4) , where z is the vertical coordinate in the physical space, ζ is the vertical coordinate in the computational space and ζ_x is the metric term $\left.\frac{\partial z}{\partial x}\right|_\zeta \Big/ \left.\frac{\partial z}{\partial \zeta}\right|_x$.

Theorem

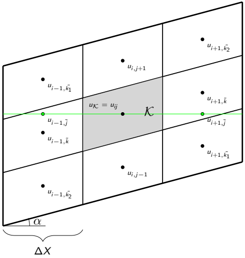
This method is stable, in the sense of von Neumann, if:

$$\frac{2K\Delta t}{\Delta x^2}\left(1 + \frac{\Delta x}{\Delta z}\frac{|\zeta_x|}{2} + \frac{\Delta x^2}{\Delta z^2}\frac{\zeta_x^2}{4}\right) \leq 1,$$

where Δt is the time step, Δx is the horizontal mesh size and Δz is the vertical mesh size in the computational space.

Improvement: Second Order Interpolation

Let us now consider the scheme which approximates horizontal diffusion using finite differences and vertical second order interpolation in place of the relation (4) in the previous scheme.



Theorem

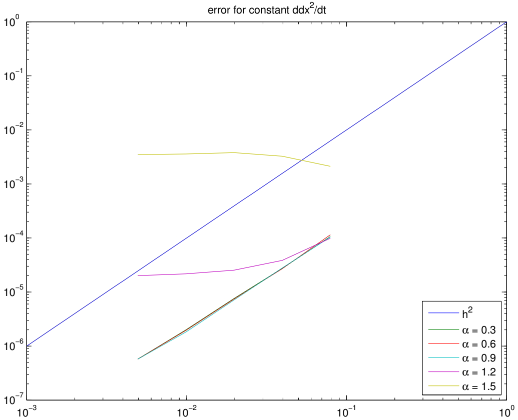
This method is stable, in the sense of von Neumann, if:

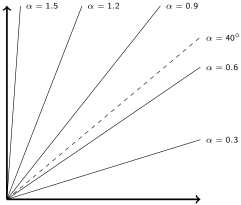
$$\frac{2K\Delta t}{\Delta x^2} \leq 1.$$

where Δt is the time step, Δx is the horizontal mesh size, and K is the diffusion coefficient.

Interpolation is an improvement but...

This method is a big improvement, as it is of order 2 for steep slopes (up to more than 40°). It is, however, not conservative.

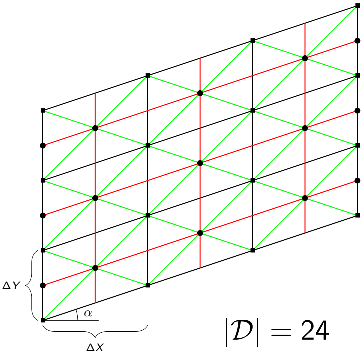




Proposed Solution: Discrete Duality Finite Volume

An additional mesh

The original unknowns are the primal nodes (q •). We introduce additional unknowns (dual nodes q^* ■), at the intersection points of the grid. We also introduce additional primal unknowns on the boundaries (q ∘). The primal mesh in black forms the primal cells ($\mathcal{K}$), the dual mesh in red forms the dual cells ($\mathcal{K}^*$) and diamond mesh ($\mathcal{D}$) is in green.



$$|\mathcal{D}| = 24$$

$$|\mathcal{K}| = 21, \quad |q| = 21$$

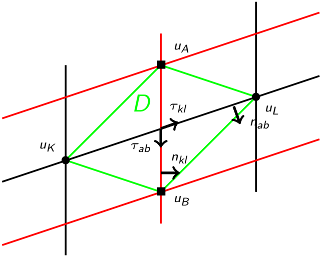
$$|\mathcal{K}^*| = 16, \quad |q^*| = 21$$

For each cell of the primal and of the dual mesh i.e. $\forall \mathcal{K}_i \in \{\mathcal{K}, \mathcal{K}^*\}$, we solve:

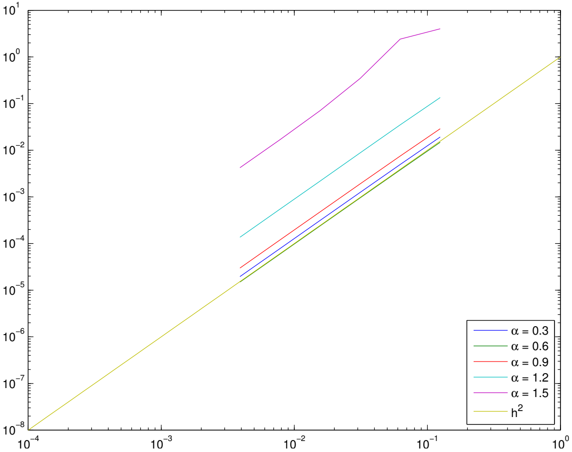
$$\int_{\mathcal{K}_i} f(x)\mathrm{d}x = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}_i}} \int_\sigma \nabla_{\mathcal{D}} u \cdot n$$

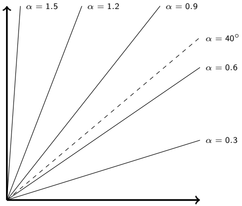
where the discrete gradient is defined as follows:

$$\nabla_{\mathcal{D}} u := \frac{u_L - u_K}{\Delta x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{u_B - u_A}{\Delta y} \begin{pmatrix} \tan\alpha \\ -1 \end{pmatrix}.$$



The implementation of the DDFV scheme on a simplified domain showed that the method is of order 2 for all $\alpha \in [0, \frac{\pi}{2}[$:





Outlook

- DDFV Prototype in MATLAB: implementation, investigate properties (stability, convergence).
- Stability theorem for DDFV.
- Implementation of DDFV in COSMO: validate, test, performance.
- Extension to anisotropic mixing tensors.